About a way of estimation of weights situated on variance ellipse and method of determination of kind of surface on which portfolio weights are located

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1. Introduction

One of basic problems of the portfolio analysis that require a solution is estimation of portfolio weights. In this article there has been shown a way of estimation of the ternary portfolio values. There are also shown the formulas that enable to determine these values in a direct way. In the part (4) there has been executed a test to adapt the quadric surfaces theorem to research a kind of surfaces on which the portfolio weights may be situated.

2. Estimation of weights situated on variance ellipse

One of the basic parameters that determine a given portfolio of securities is the portfolio variance. Assuming that the portfolio weights fulfil the dependence: $\sum_{i=1}^{3} x_i = 1$ we can receive the following form of the ternary portfolio variance[1]:

$$s^2(R_p) = x_1^2 S_1^2 + x_2^2 S_2^2 + (1 - x_1 - x_2)^2 S_3^2 + 2x_1 x_2 S_1 S_2 r_{12} + 2x_1(1 - x_1 - x_2) S_1 S_3 r_{13} + 2x_2(1 - x_1 - x_2) S_2 S_3 r_{23} = (S_1^2 - 2S_1 S_3 r_{13} + S_3^2) x_1^2 + 2(S_3^2 + S_1 S_2 r_{12} - S_1 S_3 r_{13} - S_2 S_3 r_{23}) x_1 x_2 + (S_3^2 + S_2^2 - 2S_2 S_3 r_{23}) x_2^2 + 2(S_1 S_3 r_{13} - S_3^2) x_1 + 2(S_2 S_3 r_{23} - S_3^2) x_2 + S_3^2 \quad (1)$$

where:

$s^2(R_p)$ - the portfolio variance,

x_i - participation of the purchase price of the i-th share in the portfolio purchase price (i=1, 2, 3),

S_i - the standard deviation of the return rate of the i-th share,

S_j - the standard deviation of the return rate of the j-th share,

r_{ij} - the correlation coefficient of the i-th share with the j-th share.

[1] Haugen 1993.

In the practice the correlation coefficients are calculated from the formula[2]:

where:

$$r_{ij} = \frac{\sum_{t=1}^{M}\{R_{i_t} - E(R_i)\}\{R_{j_t} - E(R_j)\}}{(M-1)S_i S_j} \tag{2}$$

where:

R_{it} - possible return rates of the i-th share (t=1, 2, ……, M),

R_{jt} - possible return rates of the j-th share (t=1, 2, ……, M),

$E(R_i)$ - the expected value of the return rate of the i-th share,

$E(R_j)$ – the expected value of the return share of the j-th share,

M – the number of all researched return rates.

In calculations used on the Stock Exchange we the most often use the following formula for the standard deviation of the i-th share[3]:

$$S_i = \sqrt{\frac{1}{M-1}\sum_{t=1}^{M}\{R_{i_t} - E(R_i)\}^2} \tag{3}$$

where:

$$E(R_i) = \frac{1}{M}\sum_{t=1}^{M}\frac{P_{ti} - P_{ti-1} + D_{ti}}{P_{ti-1}}, \quad R_{i_t} = \frac{P_{ti} - P_{ti-1} + D_{ti}}{P_{ti-1}} \quad (\text{ t }=1,2,...\text{M}) \tag{4}$$

and P_{ti}, P_{ti-1} – prices for the i-th share in the time period t and t-1 accordingly, D_{ti} – a dividend of the i-th share paid in the t-th time period.

The general form of the quadratic equation with two unknowns x_1, x_2 is as follows:

$$Ax_1^2 + 2Bx_1x_2 + Cx_2^2 + 2Dx_1 + 2Ex_2 + F = 0\,. \tag{5}$$

In our case with (1) we have:

$$A = S_1^2 - 2S_1S_3r_{13},\quad B = S_3^2 + S_1S_2r_{12} - S_1S_3r_{13} - S_2S_3r_{23},$$

$$C = S_3^2 + S_2^2 - 2S_2S_3r_{23},\ D = S_1S_3r_{13} - S_3^2,\ E = S_2S_3r_{23} - S_3^2,\ F = S_3^2 - s^2(R_p)\,. \tag{6}$$

Determining x_2 from the equation (5) we receive:

$$x_2 = -\frac{Bx_1 + E}{C} \pm \frac{1}{C}\sqrt{Mx_1^2 + 2Nx_1 + P}\,, \tag{7}$$

where: $M = B^2 - AC$, $N = BE - CD$, $P = E^2 - CF$. Calculating the discriminant of the trinomial square under the radical we receive:

[2] Tarczynski 1997.
[3] Ibidem.

$$Q = 4N^2 - 4MP = -4C\,(2BED - CD^2 - AE^2 - B^2F + ACF)\,,$$

or in the other way:

$$Q = \begin{vmatrix} -4CA & -4CB & -4CD \\ B & C & E \\ D & E & F \end{vmatrix} \quad . \tag{8}$$

As it can be proved, a kind of the curve we receive in case of (7) depends on the number M and the determinant Q, it means when, for example, M < 0 and Q > 0 we deal with the real ellipse for $\bar{x}_1 \le x_1 \le \hat{x}_1$, where $\bar{x}_1, \hat{x}_1$ - are the square roots of the trinomial square $Mx_1^{\;2} + Nx_1 + P$. Similarly M < 0 and Q < 0 give the imaginary ellipses. However in case when M > 0 and Q > 0 we receive a family of real hyperbolas, and for example M = 0 and Q > 0 give a family of real parabolas. Of course, we have not specified all the cases here. Let us try, basing on the equation (7) and the above, answer the question what conditions have to be fulfilled by the weights x_i (1 = i = 2) of the portfolio shares built on the basis of three companies, if we are anxious to have the weights situated on the real ellipse. From this that Q > 0 and M < 0 we receive $Mx_1^2 + 2Nx_1 + P = 0$ in case when:

$$\frac{-2N - \sqrt{Q}}{2M} \le x_1 \le \frac{-2N + \sqrt{Q}}{2M} \tag{9}$$

where

$$\bar{x}_1 = \frac{-2N - \sqrt{Q}}{2M},\ \hat{x}_1 = \frac{-2N + \sqrt{Q}}{2M}\ -$$

the roots of the trinomial square $Mx_1^2 + 2Nx_1 + P$.

As it is easy to see, the straight line $x_2 = \dfrac{-Bx_1 - E}{C}$ is the symmetry axis of the ellipse (7). In case when B = 0, the biggest value $x_{2\,\max}$ that can be received by the weight of shares of the second company included in the portfolio equals to:

$$x_{2\,\max} = \frac{-E}{C} + \frac{1}{C}\sqrt{\frac{-Q}{4M}} = \frac{-2E + \sqrt{\dfrac{-Q}{M}}}{2C}\,, \tag{10}$$

for $x_1 = x_o = \dfrac{\bar{x}_1 + \hat{x}_1}{2} = \dfrac{-N}{M}$ at the biggest value of the trinomial square

$Mx_1^2 + 2Nx_1 + P$ equal to $\dfrac{-Q}{4M}$. Because of the symmetry, the least value of x_2 equals to accordingly:

$$x_{2\min} = \frac{-E}{C} - \frac{1}{C}\sqrt{\frac{-Q}{4M}} = \frac{-2E - \sqrt{\frac{-Q}{M}}}{2C}. \tag{11}$$

By the force of (10 - 11) we receive:

$$\frac{-2E - \sqrt{\frac{-Q}{M}}}{2C} \le x_2 \le \frac{-2E + \sqrt{\frac{-Q}{M}}}{2C}, \tag{12}$$

and

$$\hat{x}_3 = 1 - x_{2\max} - x_0 = 1 - \frac{-2E + \sqrt{\frac{-Q}{M}}}{2C} + \frac{N}{M} = \frac{2(EM + NC) + M(2C - \sqrt{\frac{-Q}{M}})}{2MC}$$

$$\breve{x}_3 = 1 - x_{2\min} - x_0 = 1 - \frac{-2E - \sqrt{\frac{-Q}{M}}}{2C} + \frac{N}{M} = \frac{2(EM + NC) + M(2C + \sqrt{\frac{-Q}{M}})}{2MC}.$$

Similary when:

$$\hat{x}_1 = \frac{-2N - \sqrt{Q}}{2M} \to \hat{x}_2 = \frac{-B(\frac{-2N - \sqrt{Q}}{2M}) - E}{C} = \frac{2(BN - ME) + B\sqrt{Q}}{2MC} \quad \text{and}$$

$$\hat{x}_3 = 1 - \hat{x}_1 - \hat{x}_2 = 1 - (\frac{-2N - \sqrt{Q}}{2M}) - \frac{2(BN - ME) + B\sqrt{Q}}{2MC} = \frac{2M(C + E) + (C - B)(2N + \sqrt{Q})}{2MC}$$

we receive

$$\bar{x}_1 = \frac{-2N + \sqrt{Q}}{2M} \to \bar{x}_2 = \frac{2(BN - ME) - B\sqrt{Q}}{2MC}$$

and

$$\bar{x}_3 = 1 - \bar{x}_1 - \bar{x}_2 = \frac{2M(C + E) + (B - C)(\sqrt{Q} - 2N)}{2MC}.$$

The above shown way of determination of the biggest and least values of the weight x_2 is, of course, referred to only to the case when the symmetry axis of the ellipse is parallel to the x_1 axis. When we are anxious to find the least and the biggest values of the weight x_2, we have to use the differential calculus.

If $$\hat{x}_2 = \frac{-Bx_1 - E}{C} + \frac{1}{C}\sqrt{Mx_1^{\ 2} + 2Nx_1 + P},$$

then[4]:

[4] Fichtenholz 1969.

$$\hat{x}_2' = \frac{-B\sqrt{Mx_1^2 + 2Nx_1 + P} + Mx_1 + N}{C\sqrt{Mx_1^2 + 2Nx_1 + P}} . \qquad (13)$$

The equality $\hat{x}_2' = 0$ takes place when:

$$B\sqrt{Mx_1^2 + 2Nx_1 + P} = Mx_1 + N . \qquad (14)$$

Having both sides of the last equation squared and having the square roots of the trinomial square determined, we receive after necessary calculations:

$$\overset{(*)}{x}_{1/2} = \frac{-N}{M} \pm \frac{\sqrt{N^2(B^2-M)^2 + B^2M(B^2P + MP + M^2) - M^4}}{M(B^2-M)} . \qquad (15)$$

The values of received square roots have to, of course, fulfil the dependence (9) and (12) and after calculation of these roots they have to be substituted to the equation (14), just to be sure whether, by chance, we have not received the “wild” square roots resulting from squaring the equality (14). If it appears that one of the square roots fulfils the equation (14), it is necessary to determine the second derivative of the weight x_2 and check its sign for this x_1, which is the square root of the first derivative x_2'. We have the maximum if the second derivative is negative and minimum when the second derivative is positive – the function x_2 has the local minimum[5]. Using the equality (13) we receive, having the second derivative calculated:

$$\hat{x}_2'' = \left(\frac{-B}{C} + \frac{Mx_1 + N}{C\sqrt{Mx_1^2 + 2Nx_1 + P}}\right)' = \frac{MP - N^2}{C(Mx_1^2 + 2Nx_1 + P)^{\frac{3}{2}}} . \qquad (16)$$

To finish looking for the biggest and the least values of the weight x_2 situated on the ellipse of the variance given by the equality (7) it is also necessary to determine the extreme of the function:

$$\breve{x}_2 = -\frac{Bx_1 + E}{C} - \frac{1}{C}\sqrt{Mx_1^2 + 2Nx_1 + P} . \qquad (17)$$

It is easy to notice, that the first derivative can be equal to 0, it means $\breve{x}_2' = 0$ for the square roots given by the equation (15). Similarly, after calculations analogous to the above we receive:

[5] Ibidem.

$$\breve{x}_2'' = \frac{N^2 - MP}{C(Mx_1^2 + 2Nx_1 + P)^{\frac{3}{2}}} \quad . \tag{18}$$

This is also necessary to check, which will be in this case the sign of the second derivative in the points, in which the first derivative equals to 0. Recapitulating, in the general case the weight x_2 fulfils the double equality of the form:

$$\min\{\bar{x}_2, \hat{x}_2, \breve{x}_{2\,\min}\} \leq x_2 \leq \max\{\bar{x}_2, \hat{x}_2, \hat{x}_{2\,\max}\} \tag{19}$$

at the same time

$$\hat{x}_{2\,\min} = \hat{x}_2(\overset{(1)}{x}_1) \text{ or } \hat{x}_{2\,\max} = \hat{x}_2(\overset{(1)}{x}_2) \text{ when } \frac{MP - N^2}{C(M\overset{(1)}{x}_1{}^2 + 2N\overset{(1)}{x}_1 + P)^{\frac{3}{2}}} < 0$$

or

$$\frac{MP - N^2}{C(M\overset{(1)}{x}_2{}^2 + 2N\overset{(1)}{x}_2 + P)^{\frac{3}{2}}} < 0 \text{ and } \breve{x}_{2\,\min} = \breve{x}_2(\overset{(1)}{x}_1)$$

or

$$\breve{x}_{2\,\min} = \breve{x}_2(\overset{(1)}{x}_2) \text{ when } \frac{N^2 - MP}{C(M\overset{(1)}{x}_1{}^2 + 2N\overset{(1)}{x}_1 + P)^{\frac{3}{2}}} > 0 \text{ or } \frac{N^2 - MP}{C(M\overset{(1)}{x}_2{}^2 + 2N\overset{(1)}{x}_2 + P)^{\frac{3}{2}}} > 0 .$$

In this way we have received the following three weights of the considered portfolio that meet the extreme values of weights x_1 and x_2:

$$(1)\ (\hat{x}_1, \hat{x}_2, \hat{x}_3) = \left(\frac{-2N - \sqrt{Q}}{2M}, \frac{2(BN - ME) + B\sqrt{Q}}{2MC}, \frac{2M(C+E) + (C-B)(2N + \sqrt{Q})}{2MC}\right)$$

$$(2)\ (\bar{x}_1, \bar{x}_2, \bar{x}_3) = \left(\frac{-2N + \sqrt{Q}}{2M}, \frac{2(BN - ME) - B\sqrt{Q}}{2MC}, \frac{2M(C+E) + (B-C)(\sqrt{Q} - 2N)}{2MC}\right)$$

$$(3)\ (x_o, x_{2\,\max}, x_3) = \left(\frac{-N}{M}, \frac{-2E + \sqrt{\frac{-Q}{M}}}{2C}, \frac{2(EM + NC) + M(2C - \sqrt{\frac{-Q}{M}})}{2MC}\right) \text{ for B =0}$$

$$(3')\ (\hat{x}_0, \hat{x}_{2\,\max}, \hat{x}_3) = (\overset{(1)}{x}_1,\ \hat{x}_2(\overset{(1)}{x}_1),\ 1 - \overset{(1)}{x}_1 - \hat{x}_2(\overset{(1)}{x}_1)) \text{ - the general case} \tag{20}$$

$$(4)\ (x_o, x_{2\,\min}, \breve{x}_3) = \left(\frac{-N}{M}, \frac{-2E - \sqrt{\frac{-Q}{M}}}{2C}, \frac{2(EM + NC) + M(2C + \sqrt{\frac{-Q}{M}})}{2MC}\right) \text{ for B = 0}$$

$$(4')\ (\breve{x}_0, \breve{x}_{2\,\min}, \breve{x}_3) = (\overset{(1)}{x}_2,\ \breve{x}_2(\overset{(1)}{x}_2),\ 1 - \overset{(1)}{x}_2 - \breve{x}_2(\overset{(1)}{x}_2)) \text{ - the general case.}$$

Finding these four points gives a pretty well idea about the position of the real ellipse of the portfolio variance. Dependences (9), (19) determine in the general way the range of changeability of the weights x_1 and x_2 and at the same time also x_3.

The other interesting problem is determination of the weights of the given portfolio, which are situated on the real ellipse (the popular name is isoellipse), which meets a certain variance and the constant value of the weight, for example x_2. To fulfil it, it is necessary to solve the equation:

$$\overline{s}^{2}(R_p) = Ax_1^2 + 2Bx_1p + Cp^2 + 2Dx_1 + 2Ep + F \qquad (\ 21\ )$$

where: $x_2 = p = constans, \qquad \overline{s}^{2}(R_p) = constans\,.$

After detail calculations we receive the square roots of the equation (21) of the form:

$$^{(*)}x_{1/2} = \frac{-Bp - E}{C} \ {}^{+}_{-}\ \frac{\sqrt{p^2(B^2 - AC) + 2p(BE - CD) + E^2 - CF + C\overline{s}^{2}(R_p)}}{C} \qquad (\ 22\ )$$

where we can choose p only from the interval of forms (19), whilst x_1 has to fulfil the dependence (9).

3. Empirical example

On the basis of two years long week data counted from January 1994 to January 1996, there have been received, basing on the formulas (2 - 4), the following data referred to Elektrim, BRE and Universal companies, which are quoted on the Warsaw Stock Exchange: $S_1 = 0{,}104,\ S_2 = 0{,}081,\ S_3 = 0{,}1528,\ r_{12} = 0{,}69,\ r_{13} = 0{,}62,\ r_{23} = 0{,}46\,.$

Let us also assume that $s^2(R_p) = 0{,}1$. Then after calculations of the coefficients *A, B, C, D, E, F* of the equation (5) we receive the following results: $A = 0{,}014$;
$B = 0{,}014$, $C = 0{,}019$; $D = -0{,}013$; $E = -0{,}018$; $F = -0{,}077$; and
$M = B^2 - AC = -7 \cdot 10^{-5} < 0$; $N = BE - CD = -0{,}000005$, $P = E^2 - CF = 0{,}001787$,

$$Q = \begin{vmatrix} -0{,}00108 & -0{,}00108 & 0{,}001 \\ 0{,}014 & 0{,}019 & -0{,}018 \\ -0{,}013 & -0{,}018 & -0{,}077 \end{vmatrix} = 0{,}000000508 > 0\,.$$

As $Q > 0$ and $B^2 - AC < 0$ we can find that the weights x_1 and x_2, which correspond accordingly Elektrim and BRE companies, are situated on the real ellipse with the equation:

$$0{,}014x_1^2 + 0{,}028x_1x_2 + 0{,}019x_2^2 - 0{,}026x_1 - 0{,}036x_2 - 0{,}077 = 0.$$

Wanting to calculate generally at which values of the variances the weights of Elektrim and BRE companies are situated on the real ellipse, we have to solve the inequality:

$$\begin{vmatrix} -0,00108 & -0,00108 & 0.001 \\ 0,014 & 0,019 & -0,018 \\ -0,013 & -0,018 & 0,023-k \end{vmatrix} > 0\,,$$

that takes place at $s^2(R_p) = k > 0,005925$ and at the standard deviation $\sqrt{k} > 0,07697$ .At the same time, using the above received results and dependences (9), (12) and (13) we receive using programmme Mathcad 2001 the portfolio weights on the level:

$-5,162 \le x_1 \le 5,02$, $\bar{x}_2 = 4,751$, $\bar{x}_3 = 1,411$, $\hat{x}_2 = -2,751$, $\hat{x}_3 = -1,268$, $\overset{(1)}{x}_1 = -3,235$, $\overset{(1)}{x}_2 = 3,092$ and $\hat{x}_2''(\overset{(1)}{x}_1) = \hat{x}_2''(-3,235) = -0,184 < 0$ and $\breve{x}_2''(\overset{(1)}{x}_2) = \breve{x}_2''(3,092) = 18,375 > 0$.

So the function x_2 has the maximum at the point $\overset{(1)}{x}_1$ equal to

$$x_2(-3,235) = \hat{x}_{2\,\max} = 5,066 \text{ and}$$

minimum $\breve{x}_2(3,092) = \breve{x}_{2\,\min} = -3,066$.

By the force of (19) we receive :

$$\min\{-3,066;\ -2,751;\ 4,751\} \le x_2 \le \max\{-2,751;\ 4,751;\ 5,066\}\,.$$

So -3,066 = x_2 = 5,066.

Finding the detail estimation of the weights of particular companies in the portfolio facilitates an investor to find more convenient strategy of investing. Knowing in detail, in which limits the weights are changed, the investor can invest using both a short sale and without the short sale at the assumption that the variance of expected return rate equals to 0,1. Wanting to receive the detail estimations of three weights at the assumption that we know the value of the weight x_2 it is necessary to use the formula (20) taking into consideration that the sum of all weights gives 1.

4. Researching of kind of surface on which weights of quaternary portfolio are situated.

Let us assume now that we are going to invest in shares of four companies. In this case the portfolio variance has the form:

$$s^2(R_p) = x_1^2 S_1^2 + x_2^2 S_2^2 + x_3^2 S_3^2 + (1 - x_1 - x_2 - x_3)^2 S_4^2 + 2x_1x_2S_1S_2r_{12} + 2x_1x_3S_1S_3r_{13} +$$

$$2x_1(1-x_1-x_2-x_3)S_1S_4r_{14}+2x_2x_3S_2S_3r_{23}+2x_2(1-x_1-x_2-x_3)S_2S_4r_{24}+$$

$$2x_3(1\text{-}x_1-x_2-x_3)S_3S_4r_{34}=(S_4^{\ 2}-2S_1S_4r_{14}+S_1^{\ 2})x_1^{\ 2}+2(S_4^{\ 2}-S_1S_4r_{14}-S_2S_4r_{24}+S_1S_2r_{12})x_1x_2+$$

$$2(S_4^{\ 2}-S_1S_4r_{14}+S_1S_3r_{13}-S_3S_4r_{34})x_1x_3+2(S_1S_4r_{14}-S_4^{\ 2})x_1+(S_2^{\ 2}-2S_2S_4r_{24}+S_4^{\ 2})x_2^{\ 2}+$$

$$2(S_4^{\ 2}-S_2S_4r_{24}+S_2S_3r_{23}-S_3S_4r_{34})x_2x_3+2(S_2S_4r_{24}-S_4^{\ 2})x_2+(S_3^{\ 2}-2S_3S_4r_{34}+S_4^{\ 2})x_3^{\ 2}+$$

$$2(S_3S_4r_{34}-S_4^{\ 2})x_3+S_4^{\ 2}. \qquad (23)$$

Equivalently, the equation (23) can be written down in the following form:

$$Ax_1^{\ 2}+2Bx_1x_2+2Cx_1x_3+2Dx_1+Ex_2^{\ 2}+2Fx_2x_3+2Gx_2+Hx_3^{\ 2}+2Ix_3+J=0 \qquad (24)$$

where:

$$A=S_4^{\ 2}-2S_1S_4r_{14}+S_1^{\ 2};\ \ B=S_4^{\ 2}-S_1S_4r_{14}-S_2S_4r_{24}+S_1S_2r_{12};$$

$$C=S_4^{\ 2}-S_1S_4r_{14}+S_1S_3r_{13}-S_3S_4r_{34};\ \ D=S_1S_4r_{14}-S_4^{\ 2};\ \ E=S_2^{\ 2}-2S_2S_4r_{24}+S_4^{\ 2}$$

$$F=S_4^{\ 2}-S_2S_4r_{24}+S_2S_3r_{23}-S_3S_4r_{34};\ G=S_2S_4r_{24}-S_4^{\ 2};\ \ H=S_3^{\ 2}-2S_3S_4r_{34}+S_4^{\ 2}$$

$$I=S_3S_4r_{34}-S_4^{\ 2};\ \ J=S_4^{\ 2}-s^{\,2}(R_p). \qquad (25)$$

Let the companies considered in the portfolio be: (1) Elektrim, (2) BRE, (3) Universal, (4) Efekt. Then the standard deviations and correlation coefficients are as follows:

$S_1=0{,}104;\ S_2=0{,}081;\ S_3=0{,}1528;\ S_4=0{,}15;\ r_{12}=0{,}69;\ r_{13}=0{,}62;\ r_{14}=0{,}62;\ r_{23}=0{,}46;$

$r_{24}=0{,}57;\ r_{34}=0{,}75$.

To research, on which surface the weights of the first three companies are situated, the coefficients present in the equation (24) have to be determined at the very beginning. After calculations the coefficients given with dependences (25) have the following form:

A = 0,014; B = 0,012; C = 0,0054; D = -0,013; E = 0,01521; F = 0,021; G = -0,015; H = 0,011; I= -0,0053; J = -0,0775.

at the portfolio variance assumed on the level $s^2(R_p) = 0{,}1$.

As a result we have received the surface given with the equation:

$$0{,}014x_1^{\ 2}+0{,}024x_1x_2+0{,}0108x_1x_3-0{,}026x_1+0{,}01521x_2^{\ 2}+0{,}042x_2x_3-0{,}030x_2+0{,}011x_3^{\ 2}-$$

$$0{,}0106\,x_3-0{,}0775=0\,.$$

The surface of this type is a quadric surface. To research a kind of this quadric surface we have to consider signs of the following expressions[6]:

$$W=\begin{vmatrix}A & B & C\\ B & E & F\\ C & F & H\end{vmatrix},\quad V=\begin{vmatrix}A & B & C & D\\ B & E & F & G\\ C & F & H & I\\ D & G & I & J\end{vmatrix},\quad W_2=\begin{vmatrix}A & B\\ B & E\end{vmatrix}+\begin{vmatrix}E & F\\ F & H\end{vmatrix}+\begin{vmatrix}A & C\\ C & H\end{vmatrix},$$

$$W_1 = A+E+H,\qquad WW_1\ . \tag{26}$$

In our case, using the Mathcad 2001 Professional program, the following results were received: W= - 0,000003137< 0, V = 0,0000002813 > 0, W_1 = 0,04021 > 0, W_2 = - 0,000799 < 0 when: WW_1 = - 0,00000321 < 0. Because V > 0 and W_2 < 0, W > 0, WW_1 < 0, the Elektrim, BRE and Universal weights are situated on the surface of the single hull hyperboloid. Please note, that solving the inequality:

$$\begin{vmatrix}A & B & C & D\\ B & E & F & G\\ C & F & H & I\\ D & G & I & S_4^{\,2}-k\end{vmatrix}>0 \tag{27}$$

depending on the parameter $k = s^2(R_p)$ we receive the interval for the portfolio variance, at which the weights are situated on the single hull hyperboloid. In the calculated example, starting from $k = s^2(R_p) >$ 0,01086131, the weights will be situated on the surface mentioned.

Similarly for 0 < $k = s^2(R_p)$ < 0,01086131 the weights will be situated on the quadric surface with the name: double hull hyperboloid, because the determinant V < 0. Assuming the parameter, for example $x_3 = p$, the equation (24) can be converted to the form:

$0{,}014x_1^2 + 2\cdot 0{,}012x_1x_2 + 0{,}01521x_2^2 + 2(0{,}0054p - 0{,}013)x_1 + 2\cdot(0{,}021p - 0{,}015)x_2$
$+ 0{,}011p^2 - 0{,}0106p - 0{,}0775 = 0$

Because of the fact that $M = 0{,}012^2 - 0{,}014(0{,}01521) = -0{,}00006894 < 0$, the intersections of the hyperboloid are the real ellipses. Substituting the received coefficient of the last equation to the dependence (16), we receive after calculations that $Q > 0$ if and only if:

$$1{,}90890586224\cdot 10^{-7}p^2 - 1{,}0517532480\cdot 10^{-7}p + 3{,}8836240560\cdot 10^{-7} > 0$$

[6] Stark 1958.

what, as it can be easy seen, takes place for the arbitrary value of the parameter p and at the same time also for the weight x_3. Assuming now $x_2 = p =$ constant for the parameter (it means the weight level of BRE company) we receive:

$0{,}014x_1^2 + 2 \cdot 0{,}00549x_1x_3 + 2 \cdot (0{,}012p + 2 \cdot (0{,}021p - 0{,}00515)x_3 - (-0{,}01521p^2 + 0{,}03p + 0{,}0775) = 0.$

In this case $\quad M = 0{,}00549^2 - 0{,}014 \cdot 0{,}011 = -0{,}0001238 < 0.$

Whilst

$$Q = \begin{vmatrix} -4 \cdot 0{,}014 \cdot 0{,}011 & -4 \cdot 0{,}00549 \cdot 0{,}011 & -4 \cdot 0{,}011 \cdot (0{,}012p - 0{,}013) \\ 0{,}00549 & 0{,}011 & 0{,}021p - 0{,}00531 \\ 0{,}012p - 0{,}013 & 0{,}021p - 0{,}00531 & -(-0{,}01521p^2 + 0{,}030p + 0{,}0775) \end{vmatrix} > 0$$

$$1{,}3671376 \cdot 10^{-7}p^2 + 3{,}77929144 \cdot 10^{-4}p + 4{,}88177283 \cdot 10^{-7} > 0$$

what takes place for each value of the parameter p. So the intersections of considered single hull hyperboloid with planes vertical to the axis x_2 are the real ellipses and it is possible, estimating the weight shares of Elektrim and Universal in the portfolio, to use this information, which refer to the real ellipse. At the same time, fixing the portfolio variance level and the constant value of the weight x_2, it means participation of BRE shares, we can estimate in detail how Elektrim, Uniwersal and Efekt weights change, knowing additionally that $\sum_{i=1}^{4} x_i = 1$.

5. Conclusions

As it can be seen, basing on derived formulas (referred to the ellipse variance) we can relatively easy estimate the weights values of the ternary portfolio assuming that the sum of weights gives 1. However, at the constant portfolio variance (assumed in a certain interval) and fixed value of one of weights – as it is seen in the example (on the basis of the quadric surface theorem) the remaining portfolio weights are more than once situated on a certain ellipse of variances with the assumption that the sum of the portfolio weights gives 1, what allows to estimate also the weights of the quaternary portfolio.

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